# On $(f,g)$-Derivation in $BCH$-Algebra

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| ABSTRACT | ARTICLE INFO |
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| BCH-algebra is a non-empty set with the binary operation * and the constant 0, and statisfying the certain axioms. A mapping of $d$ from $X$ to itself is said to be a derivation in BCH-algebra if $d$ is both$(l,r)$-derivation and $(r,l)$-derivation in BCH-algebra, where $X$ is BCH-algebra. This article discusses the concepts of $(l,r)$-$(f,g)$-derivation, $(r,l)$-$(f,g)$-derivation, and $(f,g)$-derivation in BCH-algebra, and investigates the properties $(l,r)$-$(f,g)$-derivation, $(r,l)$-$(f,g)$-derivation and $(f,g)$-derivation in BCH-algebra. | <br><br> |

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## 1. INTRODUCTION

Hu and Li [1] introduced an algebraic structure called BCH-algebra, which is a generalization of BCK-algebra and BCI-algebra. BCH-algebra is a non-empty set $X$ with the binary operation * and the constant 0, which satisfies the axiom (BCH1) $x * x = 0$, (BCH2) $x \le y$ and $y \le x$ imply $x = y$, where $x \le y$ if and only if $x * y = 0$, and (BCH3) $(x * y) * z = (x * z) * y$ for any $x, y \in X$ [2,3].

The concepts of BCH-algebra have been discussed by researchers, for instance the concept of derivation [4-7]. The derivation in BCI-algebra was introduced by Jun and Xin [8]. In 2005 Zhan and Liu [9] introduced the notion of derivation in BCI-algebra to find the concept of $f$-derivation in BCI-algebra, where $f$ is the endomorphism in BCI-algebra [10-13]. Al-shehri [14] applied the concept of the derivation of BCI-algebra to B-algebra in 2010, then the notion was introduced by Ardekani and Davvas [15] in order to obtain a new idea, called $(f,g)$-derivation in B-algebra, where $f$ and $g$ is an endomorphism of B-algebra. In 2015, Bawazeer and Bashammakh [16] introduced the notion of derivation in BCH-algebra and fixed sets. The notion of derivation defined the concept of derivation in BCH-algebra. A mapping $d$ from $X$ to itself is called to be a derivation in BCH-algebra if $d$ is both an $(l,r)$-derivation and $(r,l)$-derivation in BCH-algebra, with $X$ is BCH-algebra.

The objective of this paper is to define $(f,g)$-derivation in BCH-algebra and investigate some of the properties of $(f,g)$-derivation in BCH-algebra.

## 2. PRELIMINARIES

In this section, the definition and some properties of BCH-algebra are given.

### 2.1. Definition 1

BCH-algebra is an algebra $(X; *, 0)$ that satisfies the following axioms [9], for all $x, y \in X$:

(BCH1)   $x * x = 0$
(BCH2)   $x \leq y$ and $y \leq x$ imply $x = y$, where $x \leq y$ if and only if $x * y = 0$
(BCH3)   $(x * y) * z = (x * z) * y$

In BCH-algebra $(X; *, 0)$ the following proporties apply, for all $x, y \in X$:

(BCH4)   $(x * (x * y)) * y = 0$
(BCH5)   $x * 0 = 0$ implies $x = 0$
(BCH6)   $0 * (x * y) = (0 * x) * (0 * y)$
(BCH7)   $x * 0 = x$
(BCH8)   $(x * y) * x = 0 * y$
(BCH9)   $x * y = 0$ implies $0 * x = 0 * y$
(BCH10)  $x * (x * y) \leq y$

Example, let $X = \{0,1,2\}$ be a set with *Cayley* table as follows,

**Table 1.** *Cayley* table for $(X; *, 0)$.

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 2 | 2 |
| 1 | 1 | 0 | 2 | 2 |
| 2 | 2 | 2 | 0 | 0 |
| 3 | 3 | 2 | 1 | 0 |

From Table 1 it can be shown that $(X; *, 0)$ satisfying all axioms BCH-algebra, so that $(X; *, 0)$ is BCI-algebra.

For a BCH-algebra $(X; *, 0)$, we denote $x \wedge y = y * (y * x)$. A mapping $f$ of a BCH-algebra $X$ into itself is called an endomorphism of $X$ if $f(x * y) = f(x) * f(y)$, for all $x, y \in X$. Note that $f(0) = 0$.

## 2.2. Definition 2

Let $(X; *, 0)$ be a BCH-algebra and $d$ is a mapping from $X$ into itself [16]. The $d$ mapping is to be $(l, r)$-derivation in $X$, if it satisfies $d(x * y) = (d(x) * y)) \wedge (x * d(y))$ for all $x, y \in X$ and $d$ called $(r, l)$-derivation in $X$ if it satisfies $d(x * y) = (x * d(y)) \wedge (d(x) * y)$ for all $x, y \in X$. Moreover, If $d$ is both an $(l, r)$-derivation and $(r, l)$-derivation, then $d$ is derivation in $X$.

## 2.3. Definition 3

Let $(X; *, 0)$ be a BCH-algebra and $d$ is a mapping from $X$ into itself [16]. The $d$ mapping is called to be regular if $d(0) = 0$.

## 3. RESULTS AND DISCUSSIONS

This section consider the main results of research, which is to define $(f, g)$-derivation in BCH-Algebra, where $f$ and $g$ are endomorphisms of BCH-Algebra [17-20]. Then, define $(f, g)$-derivation in BCH-algebra and construct its properties which are expressed in proposition form.

## 3.1. Definition $(f, g)$-Derivation in BCH-Algebra

Let $(X; *, 0)$ be a BCH-algebra and $d$ is a mapping from $X$ to itself, where $f, g$ is an endomorphism of $X$. The $d$ mapping is said to be $(l, r)$-$(f, g)$-derivation in $X$ if it satisfies $d(x * y) = (d(x) * f(y)) \wedge (g(x) * d(y))$ for all $x, y \in X$ and $d$ called $(r, l)$-$(f, g)$-derivation in $X$ if it satisfies $d(x * y) = (f(x) * d(y)) \wedge (d(x) * g(y))$ for all $x, y \in X$. Moreover, if $d$ is both an $(l, r)$-$(f, g)$-derivation and $(r, l)$-$(f, g)$-derivation, then $d$ is $(f, g)$-derivation in $X$.

Example, let $X = \{0,1,2\}$ be a set defined in Table 2,

**Tabel 2.** Tabel *Cayley* untuk $(X; *, 0)$.

| $*$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 2 |
| 1 | 1 | 0 | 2 |
| 2 | 2 | 2 | 0 |

It can be shown that $(X; *, 0)$, satisfies all BCH-algebra axioms such that $(X; *, 0)$.
Define a map $d, f, g: X \to X$ by:

$$d(x) = \begin{cases} 0 & if \ x = 0,1 \\ 2 & if \ x = 2 \end{cases}, \qquad f(x) = \begin{cases} 0 & if \ x = 0,1 \\ 2 & if \ x = 2 \end{cases}, \qquad g(x) = \begin{cases} 0 & x = 0 \\ 1 & x = 1 \\ 2 & x = 2 \end{cases}$$

So it can be proved that $f$ and $g$ are endomorphisms of $X$ and d is $(f, g)$-derivation in $X$.

### 3.1.1. Proposition 1

Let $(X; *, 0)$ be a BCH-algebra, $d$ is a self-map of $X$, where $f, g$ is an endomorphism of $X$.
1. Let $d$ be $(l, r)$-$(f, g)$-derivation in $X$. If $d$ regular then $d(x) = d(x) \wedge g(x)$, for all $x \in X$.
2. Let $d$ be $(r, l)$-$(f, g)$-derivation in $X$. $d$ regular if and only if $d(x) = f(x) \wedge d(x)$, for all $x \in X$.
   Proof,
1. Let $d$ is $(l, r)$-$(f, g)$-derivation in $X$. Since $d$ regular, then $d(0) = 0$, and by axiom (BCH7) we have:

   $d(x) = d(x * 0)$
   $d(x) = (d(x) * f(0)) \wedge (g(x) * d(0))$
   $d(x) = (d(x) * 0) \wedge (g(x) * 0)$
   $d(x) = d(x) \wedge g(x)$

   Hence, this shows that $d(x) = d(x) \wedge g(x)$, for all $x \in X$.
2. Let $d$ is $(r, l)$-$(f, g)$-derivation in $X$. Since $d$ regular, then $d(0) = 0$, and by axiom (BCH7) we have:

   $d(x) = d(x * 0)$
   $d(x) = (f(x) * d(0)) \wedge (d(x) * g(0))$
   $d(x) = (f(x) * 0) \wedge (d(x) * 0)$
   $d(x) = f(x) \wedge d(x)$

   Hence, this shows that $d(x) = f(x) \wedge d(x)$ for all $x \in X$. Conversely, let $d$ is $(r, l)$-$(f, g)$-derivation in $X$. If $d(x) = f(x) \wedge d(x)$, then for $x = 0$, we have:

   $d(0) = (f(0) \wedge d(0))$
   $d(0) = (0 \wedge d(0))$
   $d(0) = d(0) * (d(0) * 0)$
   $d(0) = d(0) * d(0)$
   $d(0) = 0$

   Hence, this shows that $d$ is regular.

### 3.1.2. Proposition 2

Let $(X; *, 0)$ be a BCH-algebra.
1. If $d$ is a $(l, r)$-$(f, g)$-derivation in $X$, then $d(x * y) \leq d(x) * f(y)$, for all $x, y \in X$.
2. If $d$ is a $(r, l)$-$(f, g)$-derivation in $X$, then $d(x * y) \leq f(x) * d(y)$, for all $x, y \in X$.
   Proof,
1. Since $d$ is a $(l, r)$-$(f, g)$-derivation in $X$ and by axiom (BCH10) we have:

$$d(x * y) = (d(x) * f(y)) \wedge (g(x) * d(y))$$
$$d(x * y) = (g(x) * d(y)) * (g(x) * d(y)) * (d(x) * f(y))$$
$$d(x * y) \leq d(x) * f(y)$$

Hence, this shows that $d(x * y) \leq d(x) * f(y)$ for all $x, y \in X$.

2. Since $d$ is a $(r, l)$-$(f, g)$-derivation in $X$ and by axiom (BCH10) we have:

$$d(x * y) = (f(x) * d(y) \wedge (d(x) * g(y))$$
$$d(x * y) = (d(x) * g(y)) * ((d(x) * g(y)) * (f(x) * d(y))$$
$$d(x * y) \leq f(x) * d(y)$$

Hence, this shows that $d(x * y) \leq f(x) * d(y)$ for all $x, y \in X$.

### 3.1.3. Proposition 3

Let $(X; *, 0)$ be a BCH-algebra.

1. If $d$ is a $(r, l)$-$(f, g)$-derivation in $X$ and $d$ regular, then $d(x) \leq f(x)$ for all $x \in X$.
   Proof,

1. Let $d$ is a $(r, l)$- $(f, g)$-derivation in $X$. Since $d$ regular, then $d(0) = 0$. By axiom (BCH7) and (BCH10), for all $x \in X$ we obtained:

$$d(x) = d(x * 0)$$
$$d(x) = (f(x) * d(0)) \wedge (d(x) * g(0))$$
$$d(x) = f(x) \wedge d(x)$$
$$d(x) = d(x) * (d(x) * f(x))$$
$$d(x) \leq f(x)$$

Hence, $d_f(x) \leq f(x)$ for all $x \in X$.

### 3.1.4. Proposition 4

Let $(X; *, 0)$ be a BCH-algebra, $d$ is a mapping from $X$ to itself.

1. Let $d$ be $(r, l)$-$(f, g)$-derivation in $X$. If $f(x) * d(x) = 0$ for all $x \in X$, then $d$ regular.
2. Let $d$ be $(l, r)$-$(f, g)$-derivation in $X$. If $d(x) * f(x) = 0$ for all $x \in X$, then $d$ regular.
   Proof,

1. Let $d$ is $(r, l)$-$(f, g)$-derivation in $X$. By axiom (BCH1) and (BCH7), for all $x \in X$ obtained:

$$d(0) = d(x * x)$$
$$d(0) = (f(x) * d(x)) \wedge (d(x) * g(x))$$
$$d(0) = 0 \wedge (d(x) * g(x))$$
$$d(0) = (d(x) * g(x)) * ((d(x) * g(x)) * 0)$$
$$d(0) = (d(x) * g(x)) * (d(x) * g(x))$$
$$d(0) = 0$$

Hence, $d(0) = 0$ then it is proved that $d$ regular.

2. Let $d$ is $(l, r)$-$(f, g)$-derivation in $X$. By axiom (BCH1) and (BCH7), for all $x \in X$ obtained:

$$d(0) = d(x * x)$$
$$d(0) = (d(x) * f(x)) \wedge (g(x) * d(x))$$
$$d(0) = 0 \wedge (g(x) * d(x))$$
$$d(0) = (g(x) * d(x)) * ((g(x) * d(x)) * 0)$$
$$d(0) = (g(x) * d(x)) * (g(x) * d(x))$$
$$d(0) = 0$$

Hence, $d(0) = 0$ then it is proved that $d$ regular.

## 3. CONCLUSIONS

This article discussed the definition of $(l, r)$ and $(r, l)$-$(f, g)$-derivation in BCH-algebra and investigate its proporties. In general, the properties of $(l, r)$ and $(r, l)$-$(f, g)$-derivation in BCH-algebra obtained for $d$ which satisfies the regular properties that is when $d(0) = 0$.

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